1. **1 The different ways to construct effective medium**
    1. 1. 1.1 The problem and the direct solution
    1. 2. 1.2 First order degenerate pertubation theory
    1. 3. 1.3 The exact way
    1. 4. 1.4 Schrieffer-Wolff transform

1. 1 The different ways to construct effective medium

1.1. 1.1 The problem and the direct solution

Consider two mass-in-mass elements, $j = 1, 2$, interacting through a soft spring of constant h. Each element has an outer mass M_j and an inner mass m_j, and a coupling spring of constant k_j. Calling the displacement of the outer mass U_j and u_j that of the inner mass, it comes that the system obeys the motion equation

$$\begin{bmatrix} k_1+h & -k_1 & -h & 0 \\ -k_1 & k_1 & 0 & 0 \\ -h & 0 & k_2+h & -k_2 \\ 0 & 0 & -k_2 & k_2 \end{bmatrix} \begin{bmatrix} U_1 \\ u_1 \\ U_2 \\ u_2 \end{bmatrix} = \omega^2 \begin{bmatrix} M_1 & & & \\ & m_1 & & \\ & & M_2 & \\ & & & m_2 \end{bmatrix} \begin{bmatrix} U_1 \\ u_1 \\ U_2 \\ u_2 \end{bmatrix} \tag{1}$$

For the unperturbated system, the equations are

$$\begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1 & 0 & 0 \\ 0 & 0 & k_2 & -k_2 \\ 0 & 0 & -k_2 & k_2 \end{bmatrix} \begin{bmatrix} U_1 \\ u_1 \\ U_2 \\ u_2 \end{bmatrix} = \omega^2 \begin{bmatrix} M_1 & & & \\ & m_1 & & \\ & & M_2 & \\ & & & m_2 \end{bmatrix} \begin{bmatrix} U_1 \\ u_1 \\ U_2 \\ u_2 \end{bmatrix} \tag{2}$$

The eigenvalues are

$$0,\ 0,\ \omega_1 = \frac{M_1 k_1 + k_1 m_1}{M_1 m_1},\ \omega_2 = \frac{M_2 k_2 + k_2 m_2}{M_2 m_2} \tag{3}$$

and the corresponding eigenvectors are

$$\begin{aligned} e_1 &= [1, 1, 0, 0]^T \\ e_2 &= [0, 0, 1, 1]^T \\ e_3 &= [k_1, -k_1 M_1/m_1, 0, 0]^T \\ e_4 &= [0, 0, k_2, -k_2 M/m_2]^T \end{aligned} \tag{4}$$

1.2. 1.2 First order degenerate pertubation theory

Assume the solution as the superposition of vector e_3 and e_4

$$u = xe_1 + ye_2 \tag{5}$$

Subsituting that into Eq. (1), we have

$$K(xe_1 + ye_2) = \omega M(xe_1 + ye_2) \tag{6}$$

Let product of e_1 and e_2 gives the equations of the coefficients x, y. The equations are

$$\begin{bmatrix} k_1^2 h + M_1^2 k_1 \omega_1^4 & -k_1 k_2 h \\ -k_1 k_2 h & k_2^2 h + M_2^2 k_2 \omega_2^4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \omega^2 \begin{bmatrix} M_1 k_1^2 \left(1 + \frac{M_1}{m_1}\right) & 0 \\ 0 & M_2 k_2^2 \left(1 + \frac{M_2}{m_2}\right) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{7}$$

If $k_1 = k_2$, $m_1 = m_2$, $M_1 = M_2$, then $\omega_1 = \omega_2$. And Eq. (7) can be reduced to

$$\begin{bmatrix} k_1^2 h & -k_1 k_2 h \\ -k_1 k_2 h & k_2^2 h \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \left(\omega^2 - \omega_0^2\right) \begin{bmatrix} M_1 k_1^2 \left(1 + \frac{M_1}{m_1}\right) & 0 \\ 0 & M_2 k_2^2 \left(1 + \frac{M_2}{m_2}\right) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{8}$$

Since this method is the first order degenerate perturbation theory, so the error exhibit $\mathcal{O}(h^2)$.

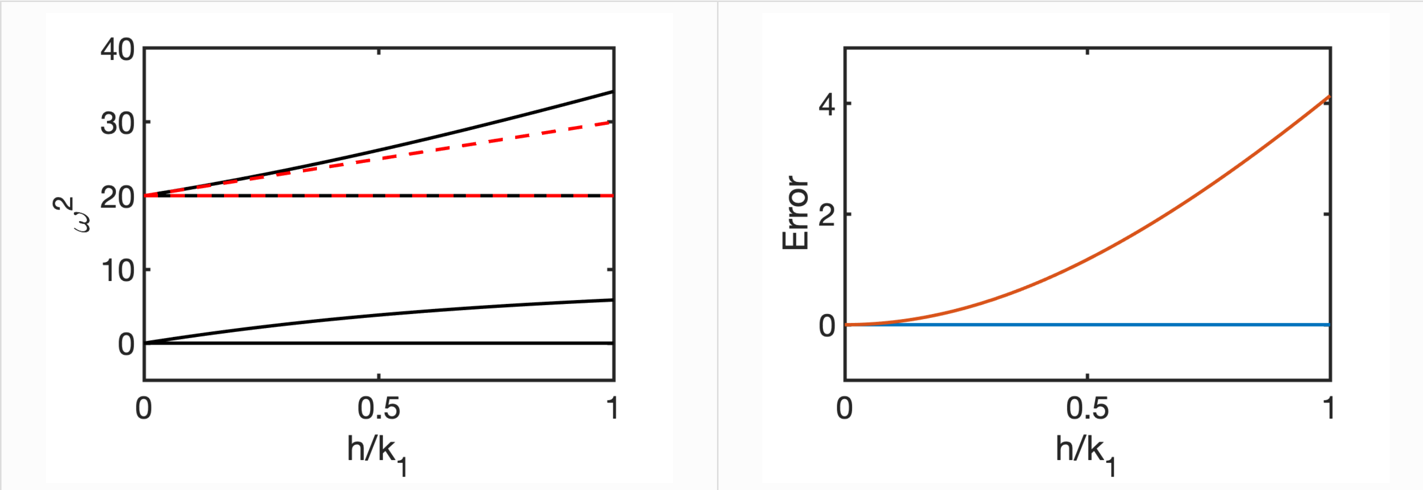

Fig. 1. Frequency spectrum for $k_1 = k_2 = 10, h = 1, m_1 = m_2 = M_1 = M_2 = 1$. (a) Black lines are from exact solution Eq. (3), and red dash lines are from first degerate perturbation theory. (b) The error is propotional to h^2.

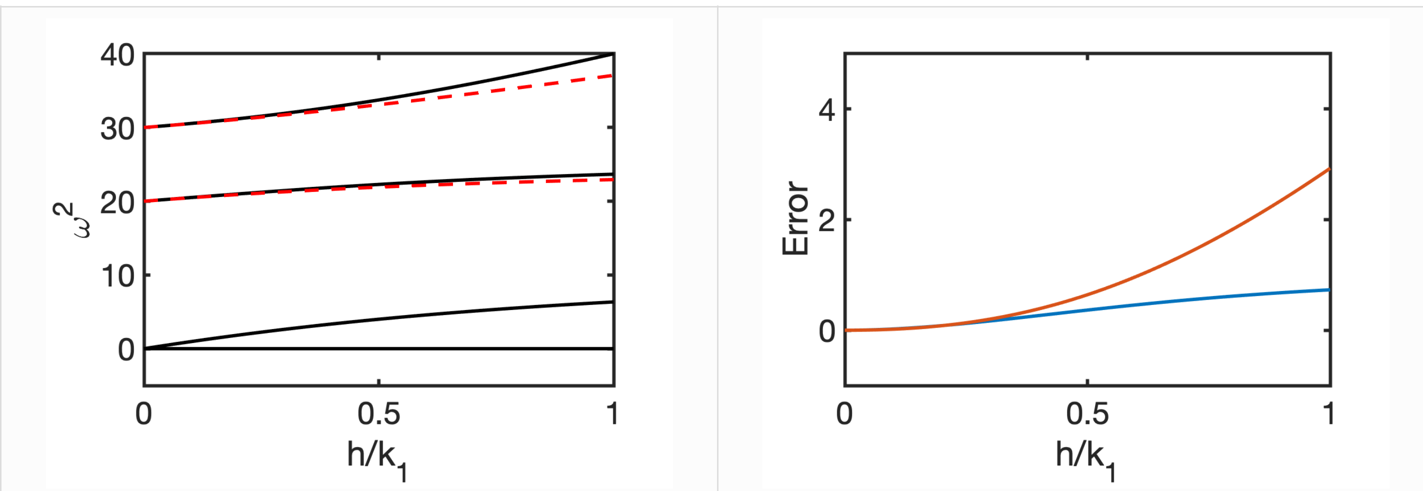

Fig. 2. Frequency spectrum for $k_1 = 10, k_2 = 15, h = 1, m_1 = m_2 = M_1 = M_2 = 1$. (a) Black lines are from exact solution Eq. (3), and red dash lines are from first degerate perturbation theory. (b) The error is propotional to h^2.

1.3. 1.3 The exact way

We transform the displacement vector to the new vector in the basis of eigenstates, or

$$\begin{bmatrix} U_1 \\ u_1 \\ U_2 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & k1 & 0 \\ 1 & 0 & -k_1 M_1/m_1 & 0 \\ 0 & 1 & 0 & -k_2 \\ 0 & 1 & 0 & -k_2 M_2/m_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \quad \text{or } u = Uv. \tag{9}$$

Substituting Eq. (9) into Eq. (1), we have

$$U^T K U v = \omega^2 U^T M U v \tag{10}$$

or

$$\begin{bmatrix} h & -h & hk_1 & -hk_2 \\ -h & h & -hk_1 & hk_2 \\ hk_1 & -hk_1 & \frac{M_1^2 k_1^3 + 2m_1 M_1 k_1^3}{m_1^2} + k_1^2 (h + k_1) & -hk_1 k_2 \\ -hk_2 & hk_2 & -hk_1 k_2 & \frac{M_2^2 k_2^3 + 2m_2 M_2 k_2^3}{m_2^2} + k_2^2 (h + k_2) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \omega^2 \begin{pmatrix} M_1 + m_1 & 0 & 0 & 0 \\ 0 & M_2 + m_2 & 0 & 0 \\ 0 & 0 & \frac{M_1 k_1^2 (M_1 + m_1)}{m_1} & 0 \\ 0 & 0 & 0 & \frac{M_2 k_2^2 (M_2 + m_2)}{m_2} \end{pmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \tag{11}$$

Define $H = K - \omega^2 M$, and rewirte $H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$ by 2 by 2 block matrices H_{ij}. Then the effective matrix of 2 v_3, v_4 is

$$H_{\text{eff}} = H_{22} - H_{21} H_{11}^{-1} H_{12}. \tag{12}$$

The spectrum can be obtained from the vanishing of determinant of Eq. (12), or

$$|H_{\mathrm{eff}}| = 0. \tag{13}$$

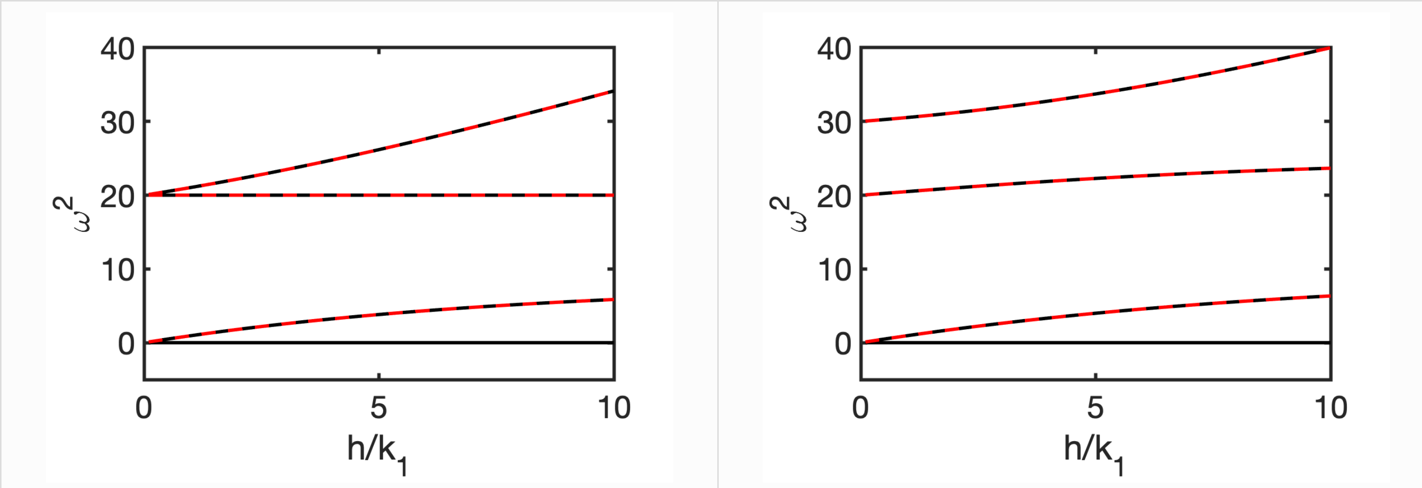

Fig. 3. Frequency spectrum for (a) $k_1 = k_2 = 10, h = 1, m_1 = m_2 = M_1 = M_2 = 1.$ and (b) $k_1 = 10, k_2 = 15, h = 1, m_1 = m_2 = M_1 = M_2 = 1.$ Black lines are from Eq. (3) and red dash lines are from Eq. (13).

1.4. 1.4 Schrieffer-Wolff transform

For simplicity, we set $m_1 = m_2 = M_1 = M_2 = 1$, so Eq. (11) can be transformed into

$$\frac{1}{2}\begin{bmatrix} h & -h & h & -h \\ -h & h & -h & h \\ h & -h & h+4k_1 & -h \\ -h & h & -h & h+4k_2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \omega^2 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \tag{14}$$

We separate Eq. (14) into the leading term and perturbative term, so we have

$$\frac{1}{2}\begin{bmatrix} h & -h & 0 & 0 \\ -h & h & 0 & 0 \\ 0 & 0 & h+4k_1 & -h \\ 0 & 0 & -h & h+4k_2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 & h & -h \\ 0 & 0 & -h & h \\ h & -h & 0 & 0 \\ -h & h & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \omega^2 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \tag{15}$$
$$\text{or} \quad Kv = \omega^2 v, \text{ where } K = K_0 + K_1.$$

Now we want to do the Schrieffer-Wolff transform to separate the low frequency subspace to the high frequency subspace. We define the unitary transform $v = \mathcal{U}w$ where $\mathcal{U} = e^S$ and S is an anti-Hermitian matrix. After the unitary transform the stiffness matrix is transformed into

$$K' = e^S K e^{-S} \tag{16}$$

By using the Baker-Campbell-Haussdorf formula , we have

$$K' = K + [S, K] + \frac{1}{2}[S, [S, K]] + \dots \tag{17}$$

or

$$K' = K_0 + K_1 + [S, K_0] + [S, K_1] - \frac{1}{2}[S, [S, K_0]] - \frac{1}{2}[S, [S, K_1]] + \dots \tag{18}$$

The Hamiltonian can be made diagonal to first order in K_1 by choosing the generator S such that

$$[K_0, S] = K_1 \tag{19}$$

This equation always has a definite solution under the assumption that hK_1 is off-diagonal in the eigenbasis of K_0. Substituting this choice in the previous transformation yields:

$$K' = K_0 + \frac{1}{2}[S, K_1] + \mathcal{O}\left(h^3\right) \tag{20}$$

where K' is block diagonalized so we finish the separation of low frequency subspace and high frequency subspace. The error after SW transform are $O\left(h^3\right)$ while it is $O\left(h^2\right)$ for first degenerate perturbation theory. Actually, the SW transform is the second degenerate pertubation theory which is not commonly used. To get S, let's diagonalize K_0 by producting a unitary transform matrix $\mathcal{U}_1$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & E_{11} & E_{12} \\ 0 & 0 & E_{21} & E_{22} \end{bmatrix} \tag{21}$$

where $K_1 E_i = d_i E_i$. And K_0 is diagonalized as

$$K_0' = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h & 0 & 0 \\ 0 & 0 & d_1 & 0 \\ 0 & 0 & 0 & d_2 \end{bmatrix} \tag{22}$$

where

$$\begin{aligned} d_1 &= \frac{h}{2} + k_1 + k_2 - \frac{\sqrt{h^2 + 4k_1^2 - 8k_1k_2 + 4k_2^2}}{2} \\ d_2 &= \frac{h}{2} + k_1 + k_2 + \frac{\sqrt{h^2 + 4k_1^2 - 8k_1k_2 + 4k_2^2}}{2}. \end{aligned} \tag{23}$$

And K_1 are transformed into

$$K_1' = \mathcal{U}_1^T K_1 \mathcal{U}_1 \tag{24}$$

and it is

$$\begin{bmatrix} 0 & 0 & K_{11}' & K_{12}' \\ 0 & 0 & K_{21}' & K_{22}' \\ K_{11}' & K_{12}' & 0 & 0 \\ K_{21}' & K_{22}' & 0 & 0 \end{bmatrix} \tag{25}$$

And the solution of S is

$$S_{ij} = \begin{cases} \frac{K_{ij}'}{d_j - d_i} & i \neq j \\ 0 & i = j \end{cases} \tag{26}$$

and then from Eq. (20), we have

$$K_d' = K_0' + \frac{1}{2}[S, K_1'] + \mathcal{O}\left(h^3\right) \tag{27}$$

Rotating back give the stiffness matrix whose low frequency subspace and high frequency subspace are separated

$$K' = \mathcal{U}_1 K_d' \mathcal{U}_1^T \tag{28}$$

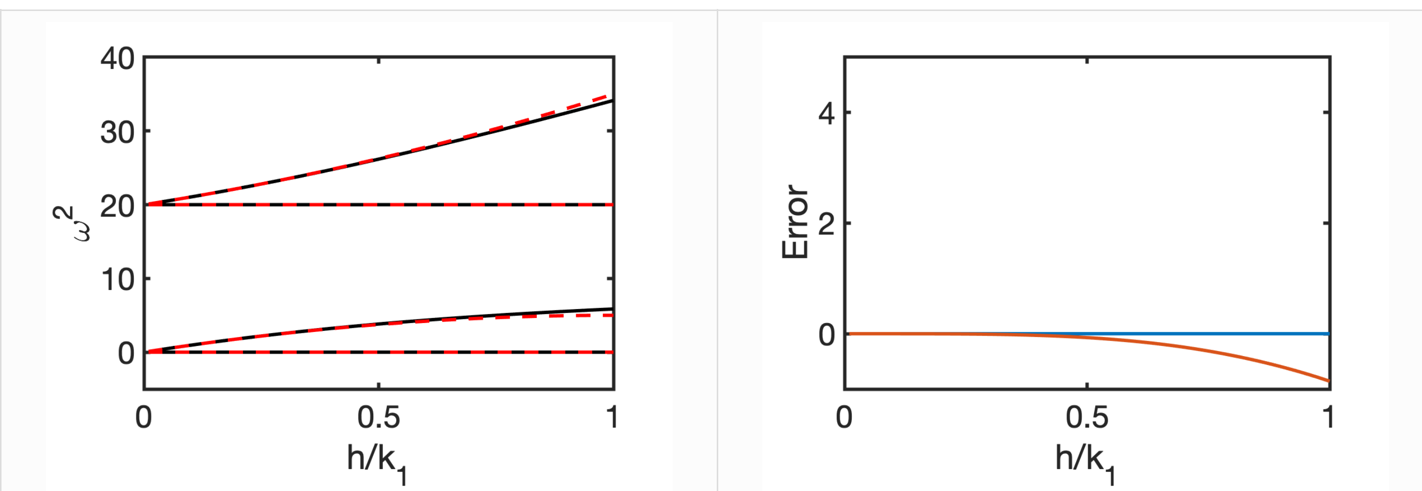

Fig. 4. Frequency spectrum for $k_1 = k_2 = 10, h = 1, m_1 = m_2 = M_1 = M_2 = 1$. (a) Black lines are from exact solution Eq. (3), and red dash lines are from SW transformation. (b) The error is propotional to h^3.

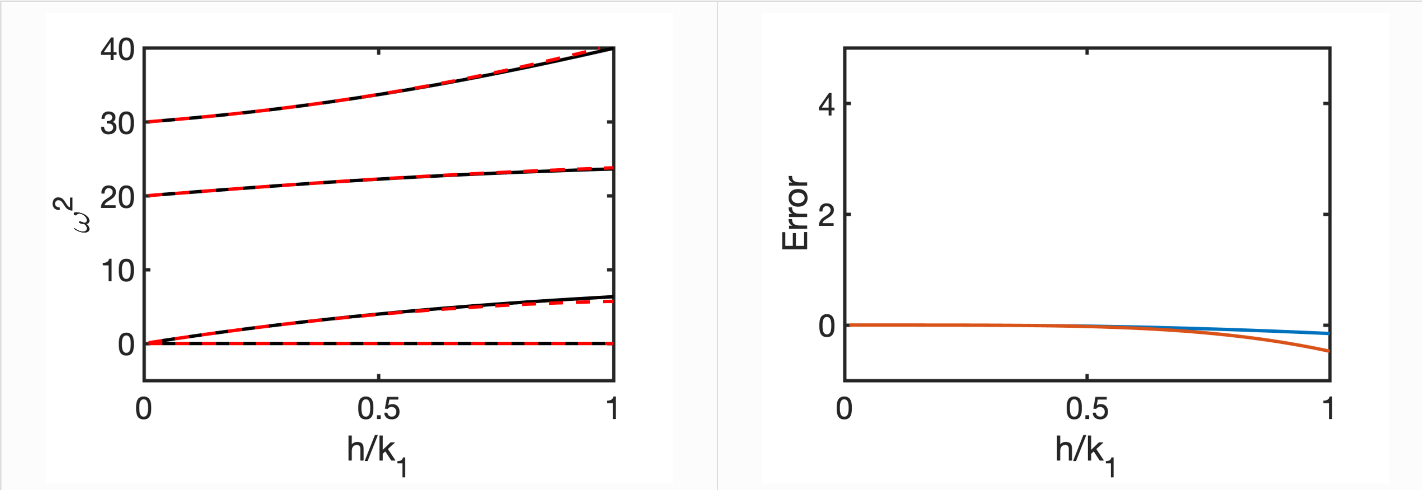

Fig. 5. Frequency spectrum for $k_1 = 10, k_2 = 15, h = 1, m_1 = m_2 = M_1 = M_2 = 1$. (a) Black lines are from exact solution Eq. (3), and red dash lines are from SW transformation. (b) The error is propotional to h^3.